

Approximating structured singular values for discrete Fourier transformation matrices

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ABSTRACT

In this article, we present the numerical computations of singular values and lower bounds of structured singular values, known as μ -values, for a family of Discrete Fourier Transform matrices. The μ -value is a well-known mathematical tool in linear control theory which speaks about the stability and instability analysis of feedback systems. The comparison of lower bounds of μ -values with the well-known MATLAB routine `mussv` is investigated.

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1. Introduction

In the present article, our main objective is to discuss the numerical approximation of Structured Singular Values (SSV) for a family of Discrete Fourier Transform matrices. The values introduced by Doyle Packard and Doyle (1993) are a well-known mathematical tool in control which discuss stability and synthesis of the linear control systems subject to certain class of uncertainties. The perturbation structures addressed by the structured singular value are very generic. These structures allows to cover all kinds of parametric perturbations which can be incorporated into the linear control system by means of both real and complex Linear Fractional Transformations LFT's, (Chen et al., 1996; Hinrichsen and Pritchard, 2005; Karow et al., 2010, 2006; Packard and Doyle, 1993; Qiu et al., 1995). Unfortunately, it's not possible to compute the exact values of SSV especially in higher dimensions computation of SSV. The reason is the fact that the exact computation is NP hard (Braatz et al., 1994). The here has been much written about the approximation of SSV; almost all of these numerical methods which are being used in practice approximated upper and lower bounds of SSV; the message from the computation of an upper bound of the SSV.

The upper bound of SSV provides sufficient conditions which guarantee the robust stability analysis of feedback systems, while on the other hand a lower bound provides sufficient conditions for guarantees the instability of the linear feedback systems in control. The well-known MATLAB function `mussv` available in the Matlab Control Toolbox approximates an upper bound of structured singular values by means of the known methodologies like as diagonal balancing technique and Linear Matrix Inequality techniques (Fan et al., 1991). The approximation of lower bound of SSV is by means of the generalization of power technique (Packard et al., 1998).

Let's consider then n -dimensional either a real (or a complex) matrix M . The matrix M could be either square or rectangular in nature. Also consider a family of block diagonal matrices Δ_G

$$\Delta_G = \{\text{Diag}(\Gamma_i, \alpha_i I_j) : \Gamma_i \in \mathbb{C}^{m_i, m_i} (\mathbb{R}^{m_i, m_i}), \alpha_i \in \mathbb{C} (\mathbb{R})\}. \quad (1)$$

In Eq. 1, I_j is an identity matrix with the dimension j , same as the dimension of the given matrix M .

Definition 1.1: Suppose that M be a n -dimensional either square or a rectangular, real (or complex) matrix and also consider a family of block diagonal matrices Δ_G . Then the SSV known as μ -value is defined as:

$$\mu_{\Delta_G} = \frac{1}{\min\{\|\Delta\|_2 : \Delta \in \Delta_G, \det(I - M\Delta) = 0\}} \quad (2)$$

In Definition 1.1, the quantity $\det(\cdot)$ is the determinant of a matrix $(I - M\Delta)$. We also consider

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the special case when the set Δ_G allows us to have only pure complex uncertainties. We denote Δ_G^* instead of Δ_G for the family of block diagonal matrices. We note that observation that $\Delta \in \Delta_G$ implies the fact that $\exp(i\varphi)\Delta \in \Delta_G$ for any scalar $\varphi \in \mathbb{R}$. This lead us with the fact that $\Delta \in \Delta_G^*$ in such a way that $\rho(M) = 1$ if and only if there exists the perturbation $\Delta' \in \Delta_G^*$ having the same unit 2-norm such that the matrix $M\Delta'$ has the maximum eigenvalue exactly equal 1, in turn this implies $\det(I - M\Delta') = 0$. The above discussion allows us to write down the following alternative expression for μ -value, that is:

$$\mu_{\Delta_G}(M) = \frac{1}{\min\{\|\Delta\|_2: \Delta \in \Delta_G^*, \rho(M\Delta) = 1\}} \quad (3)$$

In above Eq. 3, $\rho(\cdot)$ is the spectral radius of a matrix $M\Delta$.

2. μ - value based on structured ε spectral value sets

The structured epsilon spectral value set of given matrix $M \in \mathbb{C}^{n \times n}$ with respect to a perturbation level say ε is given as:

$$\Lambda_\varepsilon^{\Delta_G}(M) = \{\lambda \in \Lambda(\varepsilon M \Delta): \Delta \in \Delta_G\} \quad (4)$$

In Eq. 4, the quantity $\Lambda(\cdot)$ express the spectrum of a matrix while the admissible perturbation Δ possesses a unit 2-norm that is $\|\Delta\|_2 = 1$. For the special case when we have purely complex perturbations that is Δ_G^* , the structured spectral value set $\Lambda_\varepsilon^{\Delta_G}(M)$ is nothing but simply a disk having its center at the origin. While for a more generic case that is: of mixed complex and real uncertainties, the set

$$\Sigma_\varepsilon^{\Delta_G}(M) = \{\xi = 1 - \lambda: \lambda \in \Lambda_\varepsilon^{\Delta_G}(M)\} \quad (5)$$

allows us to express μ -value as:

$$\mu_{\Delta_G}(M) = \frac{1}{\arg \min\{0 \in \Sigma_\varepsilon^{\Delta_G}(M)\}} \quad (6)$$

For purely complex uncertainties, the underlying set Δ_G^* allows us to write down the alternative form of SSV as:

$$\mu_{\Delta_G^*}(M) = \frac{1}{\arg \min\{\max|\lambda|=1\}} \quad (7)$$

here,

$$\lambda \in \Sigma_\varepsilon^{\Delta_G}(M).$$

2.1. The mathematical problem

We consider the following optimization problem (Rehman, 2017),

$$\xi(\varepsilon) = \arg \min|\xi| \quad (8)$$

where $\xi \in \Sigma_\varepsilon^{\Delta_G}(M)$ for the fixed value of perturbation level of ε , that is $\varepsilon > 0$. From above discussion, the structured singular values $\mu_{\Delta_G}(M)$ is the reciprocal of the minimum value of perturbation level ε so that $\xi(\varepsilon) = 0$. This suggests us to give a two-level algorithm that is inner and outer algorithm: In the inner algorithm, we solve Eq. 8. While in the outer algorithm, we first vary perturbation level ε by means of some iterative method which helps to exploits the knowledge of the computation of exact derivative of an extremizer say $\Delta(\varepsilon)$ with respect to perturbation level ε . We solve the optimization problem addressed as in Eq. 4 by first solving a gradient system of Ordinary Differential Equations (ODE's). This computation only produces a local minimum of Eq. 4 which, in turn, gives an upper bound for perturbation level ε and hence as a result one obtained the lower bound for $\mu_{\Delta_G}(M)$. The purely complex uncertainties set Δ_G^* can be addressed by taking the inner algorithm to compute local optima for the maximization problem as addressed below:

$$\lambda(\varepsilon) = \arg \max|\lambda|. \quad (9)$$

Here in Eq. 9 $\lambda \in \Lambda_\varepsilon^{\Delta_G^*}(M)$ which then produces a lower bound for $\mu_{\Delta_G}(M)$.

3. Pure complex perturbations

In this section, we establish the solution of the inner problem as mentioned in Eq. 9. This includes the estimation of the quantity $\mu_{\Delta_G}(M)$ for matrix $M \in \mathbb{C}^{n,n}$ while taking into account a pure complex uncertainties set.

$$\Delta_G^* = \{\text{diag}(\alpha_1 I_1, \dots, \alpha_n I_n; \Delta_1, \dots, \Delta_F): \alpha_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{m_j \times m_j}\}. \quad (10)$$

In the following Lemma 3.1, we give the eigenvalue perturbation result in order to approximate the rate of change in the eigenvalue $\lambda(t)$.

Lemma 3.1: Let's suppose matrix family $\Omega: \mathbb{R} \rightarrow \mathbb{C}^{n,n}$ and consider that $\lambda(t)$ is one of the eigenvalue of $\Omega(t)$ which converges to a simple eigenvalue say λ_1 of $\Omega_0 = \Omega(0)$ as $t \rightarrow 0$. Then there exists eigenvectors v_0 and w_0 such that $\lambda(t)$ is analytic near $t = 0$ with

$$\dot{\lambda}(t)|_{t=0} = \frac{w_0^* \Omega_1 v_0}{w_0^* v_0}, w_0^* v_0 \neq 0, w_0^* v_0 = 1 \quad (11)$$

where, $\Omega_1 = \dot{\Omega}(0)$ and w^*, v_0^* are the right and left eigenvectors of Ω_0 associated to λ_0 , that is $(\Omega_0 - \lambda_0 I_0)v_0 = 0$ and $w_0^*(\Omega_0 - \lambda_0 I) = 0$.

Since our goal is to give a solution for the maximization problem as ad-dressed in Eq. 9. This requires the computation of a uncertainty local Δ_{local} such that $\rho(\varepsilon M \Delta_{local})$ has the maximum growth among all admissible perturbations $\Delta \in \Delta_G^*$ so that

Having a unit 2-norm that is $\|\Delta\|_2 \leq 1$. In the following we call λ_1 to be the greatest eigenvalue if $|\lambda_1|$ equals the spectral radius of the matrix $(\varepsilon M \Delta_{local})$.

Next we give the definition of a matrix valued function $\Delta(t)$ which acts as a local extremizer and maximizes the modulus of the greatest eigenvalue $\lambda_1(t)$.

Definition 3.1: A matrix values function $\Delta(t) \in \Delta_G^*$ such that $\|\Delta\|_2 \leq 1$ and $\varepsilon M \Delta$ possesses a greatest eigenvalue $\lambda(t)$ that locally maximizes the modulus of $\Sigma_\varepsilon^{\Delta G}(M)$ is known as a local extremizer (Rehman, 2017).

The following theorem provides the characterization of local extremizers $\Delta(t)$.

Theorem 3.1: Let's consider that

$$\Delta_{local} = \{\text{diag}(\alpha_i I_{r_i}; \Delta_j) \forall i = 1:S, j = 1:F\}, \|\Delta_{local}\|_2 = 1 \tag{12}$$

be a local extremizer of structured epsilon spectral value set $\Lambda_\varepsilon^{\Delta G}(M)$. Further we assume that the matrix $(\varepsilon M \Delta_{local})$ possesses the simple largest eigenvalue $\lambda_1 = |\lambda_1| e^{i\theta}$, $\theta \in R$, having both right and left eigenvectors x and y which are scaled in such a way that $k = e^{i\theta} w^* v > 0$. Partitioning these eigenvectors we have,

$$\begin{aligned} v &= (v_1^T, \dots, v_n^T, v_{n+1}^T, \dots, v_{n+F}^T)^T, \\ u &= A^* w = (u_1^T, \dots, u_n^T, u_{n+1}^T, \dots, u_{n+F}^T)^T. \end{aligned} \tag{13}$$

The size of the components v_r, u_r equals to the size of the r th block contained in local Δ_{local} . We additionally consider that

$$\begin{aligned} u_r^* v_r &\neq 0, \forall r = 1, \dots, n \\ \|u_{n+h}\|_2 \cdot \|x_{n+h}\|_2 &\neq 0 \quad \forall h = 1:F. \end{aligned} \tag{14}$$

Then, this implies that all blocks of Δ_{local} local possesses a unit2-norm, $|\alpha_r| = 1, \forall r = 1:n$ and $\|\Delta_h\|_2 = 1, \forall h = 1:F$.

By the help of the following theorem we replace full blocks Δ_{local} in a local extremizer with rank-1 matrices.

Theorem 3.2: Let's consider that

$$\Delta_{local} = \{\text{diag}(\alpha_i I_{r_i}; \Delta_j) \forall i = 1:S, j = 1:F\} \tag{16}$$

be a local extremizer and also consider that λ, x, z as partitioned in previous theorem. Assume that Eq. 14 holds, then every block Δ_h possess a singular value appears exactly equal to 1 with associated singular vectors as $q_h = \gamma_h w_{n+h} / \|w_{n+h}\|_2$ and $r_h = \gamma_h u_{n+h} / \|u_{n+h}\|_2$ for some $|\gamma_h| = 1$. Moreover, also consider that the matrix,

$$\hat{\Delta} = \{\text{diag}(\alpha_1 I_1, \dots, \alpha_n I_n; a_1 b_1^*, \dots, a_F b_F^*)\}$$

extremizer that $\rho(\varepsilon M \Delta_{local}) = \rho(\varepsilon M \hat{\Delta})$.

3.1. System of ODEs to compute extremal points of $\Lambda_\varepsilon^{\Delta G}(M)$

In order to approximate the local $\Delta(t)$ for greatest eigenvalue $|\lambda_1(t)|$, with $\lambda_1(t) \in \Lambda_\varepsilon^{\Delta G}(G)$ first we develop a matrix valued function $\Delta(t)$ that maximizes the greatest eigenvalue $\lambda_1(t)$ of $(\varepsilon M \Delta(t))$ attains the maximal local growth. Secondly we then derive a gradient system of ordinary differential equations which must satisfy the choice of this initial value matrix $\Delta(t)$ (Rehman, 2017).

3.2. The optimization problem (Rehman, 2017)

Consider that $\lambda_1(t) = |\lambda_1| e^{i\theta}$ be the simple eigenvalue having eigenvectors v, w which are normalized such that

$$\|w\|_2 = \|v\|_2 = 1, w^* v = |w^* v| e^{-i\theta} \tag{17}$$

as a result of Lemma 3.1, we get

$$\frac{d}{dt} |\lambda(t)|^2 = 2|\lambda_1| \text{Re} \left(\frac{u^* \Delta v}{e^{i\theta} w^* v} \right) = \frac{2|\lambda_1|}{|w^* v|} \text{Re}(u^* \Delta v) \tag{18}$$

where $u = M^* w$.

By considering $\Delta \in \Delta_G$, we compute the direction $\hat{\Delta} = U$ which locally maximizes the growth of the modulus of greatest eigenvalue λ_1 . From this discussion, we get

$$U = \text{diag}(w_1 I_{r_1}, \dots, w_N I_{r_N}; \Omega_1, \dots, \Omega_F) \tag{19}$$

as a solution of the maximization problem.

$$U_* = \arg \max \{ \text{Re}(u^* U x) \}$$

subject to

$$\text{Re}(\bar{\delta}_i w_i) = 0, i = 1:N$$

and

$$\text{Re}\langle \Delta_j, \Omega_j \rangle = 0, j = 1:F \tag{20}$$

The Lemma 3.2 gives the solution of the optimization problem as discussed in the Eq. 20.

Lemma 3.2: We make use of the notation as introduced earlier in the above discussion and v, u partitioned a solution U_* of the maximization problem discussed in Eq. 20 is given as (Rehman, 2017):

$$U_* = \{\text{diag}(w_1 I_{r_1}, \dots, w_N I_{r_N}; \Omega_1, \dots, \Omega_F)\} \tag{21}$$

with,

$$w_i = V_i(v_i^* u_i - \text{Re}(v_i^* u_i \bar{\delta}_i) \delta_i), \forall j = 1:N \tag{22}$$

and

$$\Omega_j = \zeta_j(u_{N+j}v_{N+j}^* - Re(\Delta_j, u_{N+j}v_{N+j}^*)\Delta_j), \forall j = 1:F. \quad (23)$$

Here in the solution U_* the coefficient $V_i > 0$ is nothing but the reciprocal of the absolute value of the right-hand side in Eq. 22 while if it is other than zero, and $V_i = 1$ else. Similarly the coefficient $\zeta_j > 0$ and is the reciprocal of the Frobenius norm of the matrix obtained on the right hand side in Eq. 23, if it is other than zero, and $\zeta_j = 1$ else.

The result obtained in Lemma 3.2 can alternatively be expressed as:

$$U_* = C_1 P_{\Delta_G^*}(zx^*) - C_2 \Delta. \quad (24)$$

In Eq. 18, $P_{\Delta_G^*}(\cdot)$ is the orthogonal projection while $C_1, C_2 \in \Delta_G^*$ are diagonal matrices where the orthogonal matrix C_1 positive.

4. The gradient system of ODEs

Lemma 4.1: allows us to consider the following differential equations on the manifold Δ_G^* .

$$\dot{\Delta} = C_1 P_{\Delta_G^*}(uv^*) - C_2 \Delta. \quad (25)$$

In Eq. 25, $v(t)$ is an eigenvector having a unit 2-norm and it is associated to a simple eigenvalue $\lambda_1(t)$ of the matrix $\varepsilon M \Delta(t)$ associated a fixed perturbation $\varepsilon > 0$. The ordinary differential Eq. 25 represents a gradient system because right-hand side is nothing but the projected gradient of $U \rightarrow Re(u^*Uv)$.

4.1. Choice of initial value matrix Δ_0 and ε_0

In two-level algorithm for approximating ε_0 we make use of the admissible perturbation obtained for the previous value ε_1 as the initial value matrix for the gradient system of ODEs. While in order to gain the locally maximal growth of $|\lambda_1(t)|$ we choose (Rehman, 2017):

$$\dot{\Delta} = CP_{\Delta_G}(wv^*) \quad (26)$$

the positive diagonal matrix C is taken in such a way that $\Delta_0 \in \Delta_G$. While on the other hand a very natural choice for ε is given as:

$$\varepsilon = \frac{1}{\hat{\mu}_{\Delta_G}(M)} \quad (27)$$

here $\hat{\mu}_{\Delta_G}(M)$ is the upper bound of μ -value which is approximated by computed by well-known MATLAB function `mussv`.

4.2. Outer algorithm

In this section, we approximate the lower bound of SSV, $\mu_{\Delta_G}(M)$ by means of outer algorithm. But we note immediately the fact that the principles remain same as discussed in the previous case, so one can treat the case of purely complex uncertainties in

detail and provide a briefer discussion on the extension to the case of mixed complex and real uncertainties (Rehman, 2017).

5. Numerical experimentation

In this section, we present a comparison of the lower bounds of μ -value approximated by `mussv` and the algorithm Rehman (2017) for a family of Discrete Fourier Transform matrices.

Example 5.1: Consider the following two dimensional Fourier Transform matrix T_2 given as,

$$T_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

consider the perturbation set as:

$$\Delta_G = \{\text{diag}(\Delta_1): \Delta \in \mathbb{C}^{2,2}\}$$

Using Matlab function `mussv`, we obtain the perturbation $\hat{\Delta}$ with $\hat{\Delta} = \begin{bmatrix} 0 & 0 \\ 0.5000 & -0.5000 \end{bmatrix}$ while $\|\hat{\Delta}\|_2 = 0.7071$.

For this example, we obtain the upper bound $\mu_{PD}^{upper} = 1.4142$ while the lower bound is approximated as $\mu_{PD}^{lower} = 1.4142$.

Now, by making use of the algorithm Rehman (2017), we obtain the perturbation $\varepsilon^* \Delta^*$ with $\Delta^* = \begin{bmatrix} -0.1464 & 0.3536 \\ 0.3536 & -0.8536 \end{bmatrix}$ while $\varepsilon^* = 0.7071$ and $\|\Delta^*\|_2 = 1$.

The same lower bound is approximated $\mu_{New}^{lower} = 1.4142$ as one approximated by MATLAB function `mussv`. Next we consider various cases for Discrete Fourier Transform matrix T_2 while taking different block diagonal structures.

Case 5.1: Consider the perturbation set as:

$$\Delta_G = \{\text{diag}(\delta_1 I_1, \delta_2 I_2): \delta_1, \delta_2 \in \mathbb{R}\}$$

Using MATLAB function `mussv`, we obtain the perturbation $\hat{\Delta}$ with $\hat{\Delta} = 1.0e + 050 * \begin{bmatrix} 5.0000 & 0 \\ 0 & 5.0000 \end{bmatrix}$ while $\|\hat{\Delta}\|_2 = 5.0000e + 050$.

For this example, we obtain the upper bound $\mu_{PD}^{upper} = 1.4142$ while the lower bound is approximated as $\mu_{PD}^{lower} = 1.3874$.

Now, by making use of the algorithm Rehman (2017), we obtain the perturbation $\varepsilon^* \Delta^*$ with $\Delta^* = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ while $\varepsilon^* = 0.7071$ and $\|\Delta^*\|_2 = 1$.

The same lower bound is approximated $\mu_{New}^{lower} = 1.4142$ as one approximated by MATLAB function `mussv`.

Case 5.2: Consider the perturbation set as:

$$\Delta_G = \{\text{diag}(\delta_1 I_1, \delta_2 I_2): \delta_1, \delta_2 \in \mathbb{C}\}$$

Using MATLAB function `musv`, we obtain the perturbation $\hat{\Delta}$ with $\hat{\Delta} = \begin{bmatrix} -0.7071 & 0 \\ 0 & -0.7071 \end{bmatrix}$ while $\|\hat{\Delta}\|_2 = 0.7071$.

For this example, we obtain the upper bound $\mu_{PD}^{upper} = 1.4142$ while the lower bound is approximated as $\mu_{PD}^{lower} = 1.4142$.

Now, by making use of the algorithm [Rehman \(2017\)](#), we obtain the perturbation $\varepsilon^* \Delta^*$ with $\Delta^* = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ while $\varepsilon^* = 0.7071$ and $\|\Delta^*\|_2 = 1$.

The same lower bound is approximated $\mu_{New}^{lower} = 1.4142$ as one approximated by MATLAB function `musv`.

Case 5.3: Consider the perturbation set as:

$$\Delta_G = \{\text{diag}(\delta_1 I_1, \delta_2 I_2) : \delta_1 \in \mathbb{R}, \delta_2 \in \mathbb{C}\}$$

$$T_3 = \begin{bmatrix} 1.0000 + 0.0000i & 1.0000 + 0.0000i & 1.0000 + 0.0000i \\ 1.0000 + 0.0000i & -0.5000 - 0.8660i & -0.5000 + 0.8660i \\ 1.0000 + 0.0000i & -0.5000 + 0.8660i & -0.5000 - 0.8660i \end{bmatrix}$$

consider the perturbation set as:

$$\Delta_G = \{\text{diag}(\Delta_1) : \Delta \in \mathbb{C}^{3,3}\}$$

$$\hat{\Delta} = \begin{bmatrix} 0.3333 - 0.0000i & 0.3333 + 0.0000i & 0.3333 + 0.0000i \\ 0.0000 + 0.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & -0.0000 + 0.0000i & -0.0000 + 0.0000i \end{bmatrix}$$

while $\|\hat{\Delta}\|_2 = 0.5774$.

For this example, we obtain the upper bound $\mu_{PD}^{upper} = 1.7321$ while the lower bound is approximated as $\mu_{PD}^{lower} = 1.6301$.

$$\Delta^* = \begin{bmatrix} 0.7887 + 0.0000i & 0.2887 + 0.0000i & 0.2887 + 0.0000i \\ 0.2887 + 0.0000i & 0.1057 + 0.0000i & 0.1057 + 0.0000i \\ 0.2887 + 0.0000i & 0.1057 + 0.0000i & 0.1057 + 0.0000i \end{bmatrix}$$

While $\varepsilon^* = 0.5774$ and $\|\Delta^*\|_2 = 1$.

The same lower bound is approximated $\mu_{New}^{lower} = 1.7321$ as one approximated by MATLAB function `musv`.

$$T_4 = \begin{bmatrix} 1.0000 + 0.0000i & 1.0000 + 0.0000i & 1.0000 + 0.0000i & 1.0000 + 0.0000i \\ 1.0000 + 0.0000i & 0.0000 - 1.0000i & -1.0000 + 0.0000i & 0.0000 + 1.0000i \\ 1.0000 + 0.0000i & -1.0000 + 0.0000i & 1.0000 + 0.0000i & 1.0000 + 0.0000i \\ 1.0000 + 0.0000i & 0.0000 + 1.0000i & -1.0000 + 0.0000i & 0.0000 - 1.0000i \end{bmatrix}$$

consider the perturbation set as:

$$\Delta_G = \{\text{diag}(\Delta_1) : \Delta \in \mathbb{C}^{4,4}\}$$

$$\hat{\Delta} = \begin{bmatrix} 0.0519 - 0.0419i & 0.0890 + 0.1058i & -0.0241 - 0.0033i & -0.0054 - 0.0606i \\ 0.1629 - 0.0291i & 0.0674 + 0.3364i & -0.0485 - 0.0359i & 0.0606 - 0.1383i \\ 0.0079 + 0.0631i & -0.1304 + 0.0192i & 0.0147 - 0.0179i & 0.0518 + 0.0260i \\ -0.0631 + 0.0079i & -0.0192 - 0.1304i & 0.0179 + 0.0147i & -0.0260 + 0.0518i \end{bmatrix}$$

While $\|\hat{\Delta}\|_2 = 0.5000$.

For this example, we obtain the upper bound $\mu_{PD}^{upper} = 2.0000$ while the lower bound is approximated as $\mu_{PD}^{lower} = 2.0000$.

Using MATLAB function `musv`, we obtain the perturbation $\hat{\Delta}$ with $\hat{\Delta} = \begin{bmatrix} -0.7071 & 0 \\ 0 & -0.7071 \end{bmatrix}$ while $\|\hat{\Delta}\|_2 = 0.7071$.

For this example, we obtain the upper bound $\mu_{PD}^{upper} = 1.4142$ while the lower bound is approximated as $\mu_{PD}^{lower} = 1.4142$.

Now, by making use of the algorithm [Rehman \(2017\)](#), we obtain the perturbation $\varepsilon^* \Delta^*$ with $\Delta^* = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ while $\varepsilon^* = 0.7071$ and $\|\Delta^*\|_2 = 1$.

The same lower bound is approximated $\mu_{New}^{lower} = 1.4142$ as one approximated by MATLAB function `musv`.

Example 5.2: Consider the following three dimensional Fourier Transform matrix T_3 given as:

Using MATLAB function `musv`, we obtain the perturbation $\hat{\Delta}$ with

Now, by making use of the algorithm [Rehman \(2017\)](#), we obtain the perturbation $\varepsilon^* \Delta^*$ with

Example 5.3: Consider the following four dimensional Fourier Transform matrix T_4 given as,

Using MATLAB function `musv`, we obtain the perturbation $\hat{\Delta}$ with

Now, by making use of the algorithm [Rehman \(2017\)](#), we obtain the perturbation $\varepsilon^* \Delta^*$ with:

$$\Delta^* = \begin{bmatrix} -0.0000 - 0.0000i & -0.0000 + 0.0000i & 0.0000 - 0.0000i & 0.0000 - 0.0000i \\ -0.0000 + 0.0000i & 0.0000 + 0.5000i & -0.0000 - 0.0000i & -0.0000 - 0.5000i \\ 0.0000 - 0.0000i & -0.0000 - 0.0000i & 0.0000 - 0.0000i & 0.0000 + 0.0000i \\ 0.0000 - 0.0000i & -0.0000 - 0.5000i & 0.0000 + 0.0000i & -0.0000 + 0.5000i \end{bmatrix}$$

While $\varepsilon^* = 0.5000$ and $\|\Delta^*\|_2 = 1$.

The same lower bound is approximated $\mu_{New}^{lower} = 2.0000$ as one approximated by MATLAB function `mussv`.

$$T_5 = \begin{bmatrix} 1 & & & & 1 \\ 1 & 0.3090 - 0.9511i & & & 1 \\ 1 & -0.8090 - 0.5878i & -0.8090 - 0.5878i & & 1 \\ 1 & -0.8090 + 0.5878i & 0.3090 - 0.9511i & & 1 \\ 1 & 0.3090 + 0.9511i & -0.8090 + 0.5878i & & 1 \end{bmatrix}$$

consider the perturbation set as:

$$\Delta_G = \{\text{diag}(\Delta_1): \Delta \in \mathbb{C}^{5,5}\}.$$

$$\hat{\Delta} = \begin{bmatrix} 0 & & & & 0 \\ 0 & -0.0431 + 0.1212i & & & 0 \\ 0 & 0.0697 + 0.1302i & 0.0697 - 0.0127i & & 0 \\ 0 & -0.0697 - 0.1302i & -0.0431 + 0.0690i & & 0 \\ 0 & 0.0431 - 0.1212i & -0.0697 + 0.0127i & & 0 \end{bmatrix}$$

While $\|\hat{\Delta}\|_2 = 0.4472$.

For this example, we obtain the upper bound $\mu_{PD}^{upper} = 2.2362$ while the lower bound is approximated as $\mu_{PD}^{lower} = 2.2362$.

Now, by making use of the algorithm Rehman (2017), we obtain the perturbation $\varepsilon^* \Delta^*$ with

$$\Delta^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.4627i & 0.1314i & -0.1314i & -0.4627i \\ 0 & 0.1314i & 0.0373i & -0.0373i & -0.1314i \\ 0 & -0.1314i & -0.0373i & 0.0373i & 0.1314i \\ 0 & -0.4627i & 0.1314i & 0.1314i & 0.4627i \end{bmatrix}$$

While $\varepsilon^* = 0.4472$ and $\|\Delta^*\|_2 = 1$.

The same lower bound is approximated $\mu_{New}^{lower} = 2.2362$ as one which is approximated by MATLAB function `mussv`.

Figs. 1-3 represent the graphical analysis of the bounds of structured singular values extracted by NewAlgo with calculated by MATLAB function `mussv`.

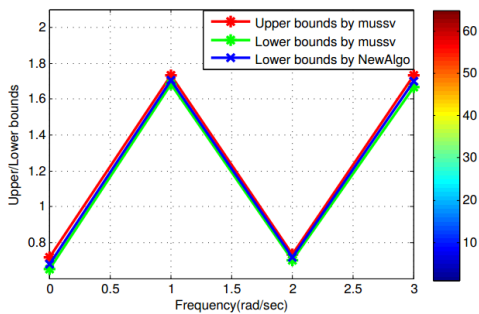


Fig. 1: Comparison of lower and upper bounds of SSV computed by `mussv` and `NewAlgo` for matrix valued function $T_2(n, w)$ for the various values of frequency $w = 1:4$ and $n = 3$

Example 5.4: Consider the following five dimensional Fourier Transform matrix T_5 given as,

Using MATLAB function `mussv`, we obtain the perturbation $\hat{\Delta}$ with

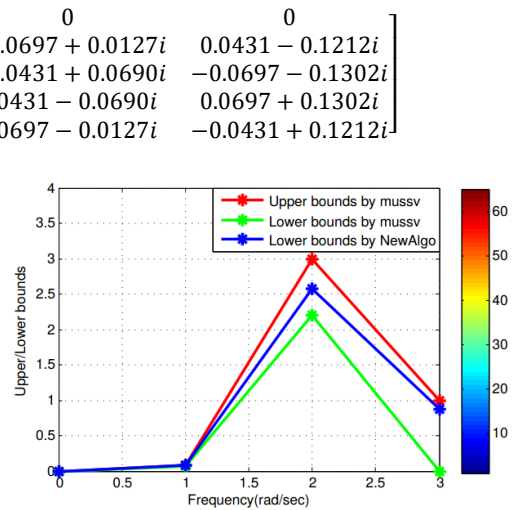


Fig. 2: Comparison of lower and upper bounds of SSV computed by `mussv` and `NewAlgo` for matrix valued function $T_3(n, w)$ for the various values of frequency $w = 1:4$ and $n = 4$

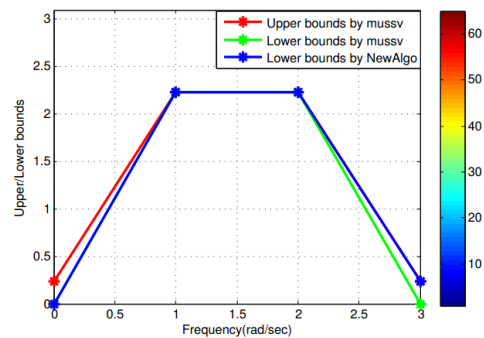


Fig. 3: Comparison of lower and upper bounds of SSV computed by `mussv` and `NewAlgo` for matrix valued function $T_3(n, w)$ for the various values of frequency $w = 1:4$ and $n = 4$

6. Conclusion

In this article we have presented the approximation of σ -values for the family of Discrete Fourier Transform matrices. Different experiments have been performed while taking into account the different Discrete Fourier Transform matrices with

various dimensions. The experimental results show how the lower bounds of SSV approximated by mussv function and the one approximated by algorithm of Rehman (2017) are related to each other's.

List of symbols

Δ_G : Family of block diagonal matrices
 ε : Perturbation level
 Δ_0 : Initial admissible perturbation
 μ : Structured singular values

References

Braatz RP, Young PM, Doyle JC, and Morari M (1994). Computational complexity of μ /calculation. IEEE Transactions on Automatic Control, 39(5): 1000-1002.
Chen J, Fan MK, and Nett CN (1996). Structured singular values with nondiagonal structures. I. Characterizations. IEEE Transactions on Automatic Control, 41(10): 1507-1511.

Fan MK, Tits AL, and Doyle JC (1991). Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. IEEE Transactions on Automatic Control, 36(1): 25-38.
Hinrichsen D and Pritchard AJ (2005). Mathematical systems theory I: modelling, state space analysis, stability and robustness. Vol. 48, Springer, Berlin, Germany.
Karow M, Kokiopoulou E, and Kressner D (2010). On the computation of structured singular values and pseudospectra. Systems and Control Letters, 59(2): 122-129.
Karow M, Kressner D, and Tisseur F (2006). Structured eigenvalue condition numbers. SIAM Journal on Matrix Analysis and Applications, 28(4): 1052-1068.
Packard A and Doyle J (1993). The complex structured singular value. Automatica, 29(1): 71-109.
Packard A, Fan MK, and Doyle J (1988). A power method for the structured singular value. In the 27th IEEE Conference on Decision and Control, IEEE, Austin, USA: 2132-2137.
Qiu L, Bernhardsson B, Rantzer A, Davison EJ, Young PM, and Doyle JC (1995). A formula for computation of the real stability radius. Automatica, 31(6): 879-890.
Rehman MU (2017). Structured singular values for Bernoulli matrices. International Journal of Advances in Applied Mathematics and Mechanics, 4(4): 41-49.